

# HILBERT SCHEMES OF POINTS FOR ASSOCIATIVE ALGEBRAS

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ABSTRACT. For a finitely generated associative algebra  $\mathcal{A}$  over a commutative ring  $k$  we construct the Hilbert scheme  $\mathbf{H}_{\mathcal{A}}^{[n]}$  which parametrizes left ideals in  $\mathcal{A}$  of codimension  $n$ .

## 1. INTRODUCTION

Fix a commutative ring  $k$ . We denote by  $F_m = k\langle x_1, \dots, x_m \rangle$  the free associative  $k$ -algebra on  $m$  generators. For a two-sided ideal  $\mathcal{R} \subset F_m$  consider the quotient  $k$ -algebra  $\mathcal{A} = F_m/\mathcal{R}$ . We are going to construct the "Hilbert scheme  $\mathbf{H}_{\mathcal{A}}^{[n]}$  of  $n$ -points" in "Spec $\mathcal{A}$ ," i.e. the moduli space of left ideals in  $\mathcal{A}$  of codimension  $n$ . As in the usual case of a commutative algebra  $\mathcal{A}$  this Hilbert scheme represents the corresponding functor. We proceed in two steps: first we construct the scheme  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ , which we call the "based Hilbert scheme". It parametrizes left ideals  $I \subset \mathcal{A}$  with a choice of a basis in the cyclic module  $\mathcal{A}/I$ . This scheme  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is quasi-affine and carries a natural free  $\mathrm{GL}_n$ -action, so that the quotient scheme is  $\mathbf{H}_{\mathcal{A}}^{[n]}$ . Thus the natural map

$$\pi : \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$$

is a Zariski locally trivial  $\mathrm{GL}_n$ -bundle. We do not use Geometric Invariant Theory to find  $\mathbf{H}_{\mathcal{A}}^{[n]}$ ; all our constructions are explicit.

We then proceed to show that  $\mathbf{H}_{\mathcal{A}}^{[n]}$  is naturally a projective scheme (over an appropriate affine scheme).

In the last section we have collected some facts about free actions of group schemes and equivariant quasi-coherent sheaves.

Hilbert schemes as ours we considered first by Van den Bergh [VdB] for the free algebra  $F_m$  over a field. He shows in particular that  $\mathbf{H}_{F_m}^{[n]}$  is smooth and constructs for it a natural cell decomposition. Later similar (or close) constructions appeared also

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in [LeB],[LeB-Se],[Rei], [En-Rei]. The case of graded (very general) Hilbert schemes was treated in [Ar-Zh].

We thank Michel Van den Bergh and Markus Reineke for pointing out to us the above references. We also thank Tony Pantev for his suggestion that our work may be related to [Ar-Zh] (at the moment we do not know if there is a relation).

## 2. CONSTRUCTION OF THE HILBERT SCHEME FOR ASSOCIATIVE ALGEBRAS

Denote by  $\text{Sch}_k$  the category of  $k$ -schemes.

Fix a finitely generated associative  $k$ -algebra  $\mathcal{A}$ . For  $X \in \text{Sch}_k$  we denote by  $\mathcal{A}_X$  the sheaf of  $\mathcal{O}_X$ -algebras which is associated to the presheaf  $\mathcal{O}_X \otimes_k \mathcal{A}$ .

### 2.1. Based Hilbert scheme $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ .

**Definition 2.1.** Denote by  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]} : \text{Sch}_k \rightarrow \text{Sets}$  the contravariant functor from the category of  $k$ -schemes to the category of sets, which is defined as follows. For a scheme  $X$  the set  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}(X)$  is the set of equivalence classes of triples  $(M, v, B)$  where  $M$  is a left  $\mathcal{A}_X$ -module which is free of rank  $n$  as  $\mathcal{O}_X$ -module,  $B \subset \Gamma(X, M)$  is a basis of the  $\mathcal{O}_X$ -module  $M$ , and  $v \in \Gamma(X, M)$  generates  $M$  as an  $\mathcal{A}_X$ -module. The equivalence relation is the obvious one.

A morphism of  $k$ -schemes  $f : Y \rightarrow X$  induces an isomorphism of sheaves of algebras

$$f^* \mathcal{A}_X = \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1} \mathcal{A}_X \rightarrow \mathcal{A}_Y$$

which gives the map of sets  $f^* : \tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}(X) \rightarrow \tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}(Y)$ .

**Theorem 2.2.** Fix  $n \geq 1$ . The functor  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}$  is represented by a quasi-affine  $k$ -scheme  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ . If the ground ring  $k$  is noetherian, then  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is of finite type over  $k$ .

*Proof.* a) Consider the polynomial ring  $k[t_{ij}^s]$  in the set of  $mn^2$  variables  $t_{ij}^s$ , where  $1 \leq s \leq m, 1 \leq i, j \leq n$ . Put  $W = \text{Spec} k[t_{ij}^s]$ . For each  $s$  denote by  $A_s = (t_{ij}^s)$  the square  $n \times n$  matrix with  $t_{ij}^s$  as the  $(i, j)$ -entry. Put  $V = \text{Spec} k[y_1, \dots, y_n]$  and let  $Y$  denote the column vector  $Y = (y_1, \dots, y_n)^t$ . It will be convenient for us to arrange the linear coordinates on the scheme  $W \times V$  as  $m$ -tuples of matrices and a column vector  $(A_1, \dots, A_m, Y)$ . For each ordered  $(n-1)$ -tuple of elements  $f_1, \dots, f_{n-1} \in F_m$  define the open subscheme  $U_{f_1 \dots f_{n-1}} \subset W \times V$  by the equation

$$(2.1) \quad D_{f_1 \dots f_{n-1}} := \det(Y, f_1(A_1, \dots, A_m)Y, \dots, f_{n-1}(A_1, \dots, A_m)Y) \neq 0$$

I.e. the coordinate ring of the affine scheme  $U_{f_1 \dots f_{n-1}}$  is the localization of the coordinate ring of  $W \times V$  at the element  $D_{f_1 \dots f_{n-1}}$ . Let  $\mathbf{U} \subset W \times V$  be the open subscheme which is the union of  $U_{f_1 \dots f_{n-1}}$ 's for all  $f_1, \dots, f_{n-1} \in F_m$ . Denote by  $Z = W \times V - \mathbf{U}$  the complementary closed subscheme which is defined by the simultaneous vanishing of the determinants  $D_{f_1 \dots f_{n-1}}$ . If  $k$  is noetherian, so is the scheme  $W \times V$ , hence in this case  $\mathbf{U}$  is the union of a finite number of  $U_{f_1 \dots f_{n-1}}$ 's.

For every relation  $r(x_1, \dots, x_m) \in \mathcal{R}$  in the algebra  $\mathcal{A}$  consider the corresponding matrix  $r(A_1, \dots, A_m)$ . Denote by  $\text{Rep}_{\mathcal{A}}^{[n]} \subset W$  the closed subscheme defined by setting  $r(A_1, \dots, A_m) = 0$  for all  $r \in \mathcal{R}$ . (So the  $S$ -points of  $\text{Rep}_{\mathcal{A}}^{[n]}$  correspond to  $k$ -algebra homomorphisms  $\mathcal{A} \rightarrow M_n(S)$  for any commutative  $k$ -algebra  $S$ .)

Finally define  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  as the scheme  $\mathbf{U} \cap (\text{Rep}_{\mathcal{A}}^{[n]} \times V)$ . Clearly  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is quasi-affine. If  $k$  is noetherian then  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is of finite type over  $k$ . We claim that it represents the functor  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}$ .

First we construct the universal triple  $(M_0, v_0, B_0)$  over  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ . Namely consider the free (left)  $k[t_{ij}^s, y_l]$ -module  $M_0 = k^n \otimes_k k[t_{ij}^s, y_l]$  of rank  $n$  with the basis  $B_0 = \{e_l \otimes 1\}$  given by the standard basis  $\{e_l\}$  of  $k^n$ , and the distinguished element  $v_0 = \sum e_l \otimes y_l$ . It will be convenient for us to think of an element  $\sum e_l \otimes \alpha_l \in M_0$  (with  $\alpha_l \in k[t_{ij}^s, y_l]$ ) as the dot product (i.e. the matrix product)

$$\mathbf{e} \bullet \alpha^t = (e_1, \dots, e_n) \bullet (\alpha_1, \dots, \alpha_n)^t$$

of the row vector  $\mathbf{e} = (e_1, \dots, e_n)$  and the column vector  $\alpha^t = (\alpha_1, \dots, \alpha_n)^t$ . In particular  $v_0 = \mathbf{e} \bullet Y$ .

The free algebra  $F_m$  acts by endomorphisms of this module  $k[t_{ij}^s, y_l]$ -module  $M_0$  via the matrices  $A_s$ . Namely in the above notation

$$x_s(\mathbf{e} \bullet \alpha^t) = \mathbf{e} \bullet A_s \alpha^t.$$

(Here  $A_s \alpha^t$  is the matrix product of the square matrix  $A_s$  with the column vector  $\alpha^t$ .) If we restrict this module to the closed subscheme  $\text{Rep}_{\mathcal{A}}^{[n]} \times V$ , then the  $F_m$ -action descends to the  $\mathcal{A}$ -action. Finally if we further restrict to the open subset  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ , then the element  $v_0$  will be a generator of this  $\mathcal{A}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}}$ -module. We denote again by  $(M_0, v_0, B_0)$  the resulting triple over  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ .

This universal triple over  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  defines an obvious morphism of the functor represented by the scheme  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  to the functor  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}$ . To show that this morphism is an isomorphism we will construct the inverse map. Namely let  $X$  be a  $k$ -scheme

with a triple  $(M, v, B)$ . Each generator  $x_s$  of the algebra  $\mathcal{A}$  acts on the basis  $B$  by an  $n \times n$ -matrix with values in  $\Gamma(X, \mathcal{O}_X)$ . Thus to each variable  $t_{ij}^s$  we associate a global function on  $X$ , which is the  $(i, j)$ -entry of this matrix. Now express the element  $v$  as a  $\Gamma(X, \mathcal{O}_X)$ -linear combination  $v = \sum \gamma_l b_l$  of the vectors  $b_l$  in  $B$  and associate to  $y_l$  the element  $\gamma_l$ . One checks that this association defines a morphism of schemes  $f : X \rightarrow \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  so that the pullback of the universal triple is isomorphic to  $(M, v, B)$ . This defines the inverse morphism of functors and proves the theorem.  $\square$

**Example 2.3.** Let  $n = 1$ . Then  $\text{Rep}_{\mathcal{A}}^{[1]} = \text{Spec} \mathcal{A}_{ab}$ , where  $\mathcal{A}_{ab} = \mathcal{A}/(\mathcal{A}[\mathcal{A}, \mathcal{A}]\mathcal{A})$  is the abelianization of the algebra  $\mathcal{A}$ ;  $W \times V = \text{Spec} k[t^1, \dots, t^m, y]$  and  $\mathbf{U} = \{y \neq 0\}$ . So  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[1]} = \text{Spec} \mathcal{A}_{ab} \times \mathbb{G}_m$ .

**Proposition 2.4.** In the above notation let  $k \rightarrow k'$  be a homomorphism of commutative algebras. Consider the algebra  $\mathcal{A}' = \mathcal{A} \otimes_k k'$ . Then for any  $n \geq 1$

$$\tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]} = \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \times_k k'.$$

*Proof.* For the purpose of this proof denote by  $X'$  the  $k'$ -scheme obtained from a  $k$ -scheme  $X$  by extension of scalars from  $k$  to  $k'$ . We use the notation of the proof of Theorem 2.2.

First it is clear that  $\text{Rep}_{\mathcal{A}'}^{[n]} = (\text{Rep}_{\mathcal{A}}^{[n]})'$ . Hence  $(\text{Rep}_{\mathcal{A}}^{[n]} \times_k V)' = \text{Rep}_{\mathcal{A}'}^{[n]} \times_{k'} V'$ .

Next recall that the  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is the open subscheme of  $\text{Rep}_{\mathcal{A}}^{[n]} \times_k V$  which is the complement of the closed subscheme  $Z$  defined by the ideal generated by elements  $D_{f_1 \dots f_{n-1}}$ . Extending the scalars to  $k'$  we see that the closed subscheme  $Z' \subset (\text{Rep}_{\mathcal{A}}^{[n]} \times_k V)' = \text{Rep}_{\mathcal{A}'}^{[n]} \times_{k'} V'$  is the complement of  $\tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]}$  (since the determinants are multi-linear in columns). Hence it follows that  $(\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]})' = \tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]}$ .  $\square$

**Remark 2.5.** Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a surjective homomorphism of (finitely generated associative)  $k$ -algebras. This induces an obvious morphism of functors  $\phi_* : \tilde{\mathbf{M}}_{\mathcal{B}}^{[n]} \rightarrow \tilde{\mathbf{M}}_{\mathcal{A}}^{[n]}$  by restriction of scalars from  $\mathcal{B}$  to  $\mathcal{A}$ . This morphism of functors corresponds to a morphism of representing schemes  $\phi_* : \tilde{\mathbf{H}}_{\mathcal{B}}^{[n]} \rightarrow \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ . It follows easily from the description the scheme  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  in Theorem 2.2 that  $\phi_*$  is a closed embedding.

## 2.2. Hilbert scheme $\mathbf{H}_{\mathcal{A}}^{[n]}$ .

**Definition 2.6.** Denote by  $\mathbf{M}_{\mathcal{A}}^{[n]} : \text{Sch}_k \rightarrow \text{Sets}$  the contravariant functor from the category of  $k$ -schemes to the category of sets, which is defined as follows. For a scheme  $X$  the set  $\mathbf{M}_{\mathcal{A}}^{[n]}(X)$  is the set of equivalence classes of pairs  $(M, v)$ , where  $M$

is a left  $\mathcal{A}_X$ -module which is locally free of rank  $n$  as  $\mathcal{O}_X$ -module and  $v \in \Gamma(X, M)$  generates  $M$  as an  $\mathcal{A}_X$ -module. The equivalence relation is the obvious one.

A morphism of  $k$ -schemes  $f : Y \rightarrow X$  induces an isomorphism of sheaves of algebras

$$f^* \mathcal{A}_X = \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1} \mathcal{A}_X \rightarrow \mathcal{A}_Y$$

which gives the map of sets  $f^* : \mathbf{M}_{\mathcal{A}}^{[n]}(X) \rightarrow \mathbf{M}_{\mathcal{A}}^{[n]}(Y)$ .

Note that we have the obvious forgetful morphism of functors  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{M}_{\mathcal{A}}^{[n]}$  which maps a triple  $(M, v, B)$  to the pair  $(M, v)$ .

**Theorem 2.7.** Fix  $n \geq 1$ . The functor  $\mathbf{M}_{\mathcal{A}}^{[n]}$  is represented by a  $k$ -scheme  $\mathbf{H}_{\mathcal{A}}^{[n]}$ . If  $k$  is noetherian then  $\mathbf{H}_{\mathcal{A}}^{[n]}$  is of finite type over  $k$ . The canonical morphism  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$  is a Zariski locally trivial principal  $\mathrm{GL}_n$ -bundle (4.1).

*Proof.* We use the notation as in the proof of Theorem 2.2. The  $k$ -group scheme  $\mathrm{GL}_n$  acts on the affine scheme  $W \times V = \mathrm{Spec} k[t_{ij}^s, y_l]$  in a natural way. Namely if we arrange as above the linear coordinates on the scheme  $W \times V$  as  $m$ -tuples of matrices and a column vector  $(A_1, \dots, A_m, Y)$  then a matrix  $g \in \mathrm{GL}_n$  acts by the formula

$$g(A_1, \dots, A_m, Y) = (gA_1g^{-1}, \dots, gA_mg^{-1}, gY)$$

The closed subscheme  $\mathrm{Rep}_{\mathcal{A}}^{[n]} \times V$  is invariant under this action. Also each affine open subscheme  $U_{f_1 \dots f_{n-1}} \subset W \times V$  is  $\mathrm{GL}_n$ -invariant. Hence  $\mathbf{U} \subset W \times V$  is invariant, and therefore  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is also  $\mathrm{GL}_n$ -invariant.

Denote  $U_{f_1 \dots f_{n-1}}^{\mathcal{A}} = U_{f_1 \dots f_{n-1}} \cap \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  and consider the  $\mathrm{GL}_n$ -equivariant map  $\psi_{f_1 \dots f_{n-1}} : U_{f_1 \dots f_{n-1}}^{\mathcal{A}} \rightarrow \mathrm{GL}_n$  which (in the above matrix notation) maps a point  $(a^1, \dots, a^m, b)$  to the matrix  $(b, f_1(a^1, \dots, a^m)b, \dots, f_{n-1}(a^1, \dots, a^m)b)$ . It follows from Proposition 4.3 below that there exists a categorical quotient

$$\pi : \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$$

which is a Zariski locally trivial principal  $\mathrm{GL}_n$ -bundle. It remains to prove that the scheme  $\mathbf{H}_{\mathcal{A}}^{[n]}$  represents the functor  $\mathbf{M}_{\mathcal{A}}^{[n]}$  and that  $\pi$  corresponds to the forgetful morphism of functors  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{M}_{\mathcal{A}}^{[n]}$ .

First let us lift the  $\mathrm{GL}_n$ -action to the free  $\mathcal{O}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}}$ -module  $M_0 = k^n \otimes_k \mathcal{O}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}}$ . Recall that we represent an element of  $M_0$  as the dot product

$$\mathbf{e} \bullet \alpha^t = (e_1, \dots, e_n) \bullet (\alpha_1, \dots, \alpha_n)^t$$

of the row vector  $\mathbf{e} = (e_1, \dots, e_n)$  and the column vector  $\alpha^t = (\alpha_1, \dots, \alpha_n)^t$ , where  $\alpha_l \in k[t_{ij}^s, y_l]$ . Then for  $g \in \mathrm{GL}_n$  we define

$$g(\mathbf{e} \bullet \alpha^t) := \mathbf{e} g^{-1} \bullet g(\alpha^t),$$

where  $\mathbf{e} \bullet g^{-1}$  is the matrix product and  $g(\alpha^t) = (g(\alpha_1), \dots, g(\alpha_n))^t$ . For  $\beta \in k[t_{ij}^s, y_l]$  and  $g \in \mathrm{GL}_n$  we have

$$g(\beta(\mathbf{e} \bullet \alpha^t)) = g(\beta)g(\mathbf{e} \bullet \alpha^t)$$

so that  $M_0$  is a  $\mathrm{GL}_n$ -equivariant vector bundle on  $\mathcal{O}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}}$ .

It follows from Proposition 4.5 below that there exists on  $\mathbf{H}_{\mathcal{A}}^{[n]}$  a unique (up to an isomorphism) locally free sheaf  $\mathcal{M}$  with an isomorphism  $\pi^* \mathcal{M} \simeq M_0$ . This sheaf  $\mathcal{M}$  is isomorphic to  $(\pi_* M_0)^{\mathrm{GL}_n}$  - the subsheaf of  $\pi_* M_0$  consisting of  $\mathrm{GL}_n$ -invariant sections, i.e.  $\mathrm{GL}_n$ -invariant sections of  $M_0$  descend to sections of  $\mathcal{M}$ . The next lemma provides many sections of the sheaf  $\mathcal{M}$ .

**Lemma 2.8.** (a) *The element  $v_0 = \mathbf{e} \bullet Y$  is  $\mathrm{GL}_n$ -invariant.*

(b) *More generally, for any  $f(x_1, \dots, x_m) \in \mathcal{A}$  the element  $f(x_1, \dots, x_m)(v_0)$  is  $\mathrm{GL}_n$ -invariant.*

(c) *The invariant sections  $f(x_1, \dots, x_m)(v_0)$  generate the sheaf  $\mathcal{M}$  as an  $\mathcal{O}_{\mathbf{H}_{\mathcal{A}}^{[n]}}$ -module.*

(d) *The action of operators  $x_i \in \mathcal{A}$  on  $M_0$  descends to an action on  $\mathcal{M}$ , which makes it an  $\mathcal{A}_{\mathbf{H}_{\mathcal{A}}^{[n]}}$ -module.*

(e) *Denote by  $v_u \in \Gamma(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{M})$  the element  $v_0$  considered as a global section of  $\mathcal{M}$ . Then  $v_u$  is a generator of  $\mathcal{M}$  as an  $\mathcal{A}_{\mathbf{H}_{\mathcal{A}}^{[n]}}$ -module.*

*Proof.* (a) and (b). Recall that  $f(x_1, \dots, x_m)(\mathbf{e} \bullet Y) = \mathbf{e} \bullet f(A_1, \dots, A_m)Y$ , where  $f(A_1, \dots, A_m)$  is a square matrix with entries in  $k[t_{ij}^s, y_l]$  and  $f(A_1, \dots, A_m)Y$  is the product of matrices. Then by definition of the  $\mathrm{GL}_n$ -action on  $k[t_{ij}^s, y_l]$  and on  $M_0$  we have

$$g(f(x_1, \dots, x_m)(\mathbf{e} \bullet Y)) = \mathbf{e} g^{-1} \bullet g(f(A_1, \dots, A_m)Y) = \mathbf{e} g^{-1} \bullet g f(A_1, \dots, A_m) g^{-1} gY,$$

where  $g f(A_1, \dots, A_m) g^{-1} gY$  is the matrix product of matrices  $g, f(A_1, \dots, A_m), g^{-1}, g, Y$ . So

$$g(f(A_1, \dots, A_m)(\mathbf{e} \bullet Y)) = \mathbf{e} \bullet f(A_1, \dots, A_m)Y = f(x_1, \dots, x_m)(\mathbf{e} \bullet Y).$$

c) Choose  $f_1, \dots, f_{n-1} \in \mathcal{A}$  and consider the corresponding affine open subset  $U_{f_1, \dots, f_{n-1}}^{\mathcal{A}} \subset \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  as above. By Lemma 4.2 there exists a  $\mathrm{GL}_n$ -equivariant isomorphism  $U_{f_1, \dots, f_{n-1}}^{\mathcal{A}} \simeq \mathrm{GL}_n \times \overline{U}$  for an open subset  $\overline{U} \subset \mathbf{H}_{\mathcal{A}}^{[n]}$ . The invariant sections

$$\mathbf{e} \bullet Y, \mathbf{e} \bullet f_1(A_1, \dots, A_m)Y, \dots, \mathbf{e} \bullet f_{n-1}(A_1, \dots, A_m)Y$$

form a basis of the restriction of  $M_0$  to  $U_{f_1, \dots, f_{n-1}}$ . It follows that they also form a basis for the restriction of  $\mathcal{M}$  to  $\overline{U}$ .

d) Recall that the  $\mathcal{A}$ -module structure on  $M_0$  is given by the formula

$$x_s(\mathbf{e} \bullet \alpha^t) = \mathbf{e} \bullet A_s \alpha^t$$

Hence  $x_i$  maps an invariant section  $\mathbf{e} \bullet f(A_1, \dots, A_m)Y$  to an invariant section  $\mathbf{e} \bullet A_s f(A_1, \dots, A_m)Y$ . By c) this implies that  $x_i$  preserves the space of invariant sections. So the action of  $x_i$  descends to  $\mathcal{M}$ , which makes it an  $\mathcal{A}_{\mathbf{H}_{\mathcal{A}}^{[n]}}$ -module.

e) This is now clear from the proof of c) and d).  $\square$

So  $(\mathcal{M}, v_u)$  is a pair on  $\mathbf{H}_{\mathcal{A}}^{[n]}$  as in Definition 2.6 and clearly the pair  $(M_0, v_0)$  on  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  is the pullback of  $(\mathcal{M}, v_u)$  under the map  $\pi$ . It remains to show that  $\mathbf{H}_{\mathcal{A}}^{[n]}$  represents the functor  $\mathbf{M}_{\mathcal{A}}^{[n]}$ . The pair  $(\mathcal{M}, v_u)$  defines a morphism from the functor represented by  $\mathbf{H}_{\mathcal{A}}^{[n]}$  to the functor  $\mathbf{M}_{\mathcal{A}}^{[n]}$ . Let us construct the inverse morphism.

Let  $Y$  be a scheme and  $(N, w)$  be a pair on  $Y$  representing an element of  $\mathbf{M}_{\mathcal{A}}^{[n]}(Y)$ . Choose an open covering  $\{V_i\}$  of  $Y$  such that for each  $i$  the restriction  $N_i := N|_{V_i}$  is a free  $\mathcal{O}_{V_i}$  module of rank  $n$ . Choose a basis  $B_i$  of  $N_i$  and let  $w_i \in N_i$  be the restriction of  $w$ . Then by Theorem 2.2 the triple  $(N_i, w_i, B_i)$  defines a canonical map  $\phi_i : V_i \rightarrow \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$ . On the intersection  $V_i \cap V_j$  there is a matrix  $g \in \mathrm{GL}_n(\mathcal{O}_{V_i \cap V_j})$  such that  $B_j = B_i \cdot g^{-1}$  (as usual we think of a basis as a row vector). This matrix defines a map  $g : V_i \cap V_j \rightarrow \mathrm{GL}_n$  such that  $g \cdot \phi_i = \phi_j$ . This shows that the compositions  $\pi \cdot \phi_i$  and  $\pi \cdot \phi_j$  agree on  $V_i \cap V_j$ , hence we obtain the map  $\psi : Y \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$  such that the pullback of the pair  $(\mathcal{M}, v_u)$  is isomorphic to  $(N, w)$ . This proves that the functor  $\mathbf{M}_{\mathcal{A}}^{[n]}$  is represented by  $\mathbf{H}_{\mathcal{A}}^{[n]}$ .

We also proved that the morphism  $\pi : \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$  is a Zariski locally trivial principal  $\mathrm{GL}_n$ -bundle and it corresponds to the canonical morphism of functors  $\tilde{\mathbf{M}}_{\mathcal{A}}^{[n]} \rightarrow \mathbf{M}_{\mathcal{A}}^{[n]}$ . This proves the theorem.  $\square$

**Example 2.9.** Let  $n = 1$ . We have seen in Example 2.3 above that  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[1]} = \mathrm{Spec} \mathcal{A}_{ab} \times \mathbb{G}_m$ . Then  $\mathbf{H}_{\mathcal{A}}^{[1]} = \mathrm{Spec} \mathcal{A}_{ab}$  and  $\pi$  is the projection.

**Proposition 2.10.** *In the above notation let  $k \rightarrow k'$  be a homomorphism of commutative algebras. Consider the algebra  $\mathcal{A}' = \mathcal{A} \otimes_k k'$ . Then for any  $n \geq 1$*

$$\mathbf{H}_{\mathcal{A}'}^{[n]} = \mathbf{H}_{\mathcal{A}}^{[n]} \times_k k'.$$

*Proof.* By Proposition 2.4 we have  $\tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]} = \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \times_k k'$ . Also the  $\mathrm{GL}_n \times_k k'$ -action on  $\tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]}$  is obtained from the  $\mathrm{GL}_n$ -action on  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  by extending scalars from  $k$  to  $k'$ . Since  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  (resp.  $\tilde{\mathbf{H}}_{\mathcal{A}'}^{[n]}$ ) is a Zariski locally trivial principal  $\mathrm{GL}_n$ -bundle over  $\mathbf{H}_{\mathcal{A}}^{[n]}$  (resp.  $\mathrm{GL}_n \times_k k'$ -bundle over  $\mathbf{H}_{\mathcal{A}'}^{[n]}$ ) we conclude that the scheme  $\mathbf{H}_{\mathcal{A}'}^{[n]}$  is obtained from  $\mathbf{H}_{\mathcal{A}}^{[n]}$  by extending scalars from  $k$  to  $k'$ .  $\square$

**Remark 2.11.** *Let  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  be a surjective homomorphism of (finitely generated associative)  $k$ -algebras. This induces an obvious morphism of functors  $\phi_* : \mathbf{M}_{\mathcal{B}}^{[n]} \rightarrow \mathbf{M}_{\mathcal{A}}^{[n]}$  by restriction of scalars from  $\mathcal{B}$  to  $\mathcal{A}$ . This morphism of functors corresponds to a morphism of representing schemes  $\phi_* : \mathbf{H}_{\mathcal{B}}^{[n]} \rightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$ . It follows easily from the description of the scheme  $\mathbf{H}^{[n]}$  in Theorem 2.7 and from Remark 2.5 that  $\phi_*$  is a closed embedding.*

### 3. PROJECTIVITY OF $\mathbf{H}_{\mathcal{A}}^{[n]}$

The universal bundle  $\mathcal{M}$  defines a cohomology class  $\Delta \in H^1(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathrm{GL}_n)$ . We can explicitly describe a 1-cocycle corresponding to  $\Delta$ . Namely recall that  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  has an affine open covering consisting of  $\mathrm{GL}_n$ -invariant subsets  $U_{f_1 \dots f_{n-1}}^{\mathcal{A}}$ . Denote the collection  $f_1, \dots, f_{n-1} \subset \mathcal{A}$  by  $\mathbf{f}$  and put  $U_{\mathbf{f}} = U_{f_1 \dots f_{n-1}}^{\mathcal{A}}$ ,  $\overline{U}_{\mathbf{f}} = \pi(U_{\mathbf{f}})$ ,  $M_{\mathbf{f}} = (Y, f_1(A_1, \dots, A_m)Y, \dots, f_{n-1}(A_1, \dots, A_m)Y)$ ,  $D_{\mathbf{f}} = D_{f_1 \dots f_{n-1}} = \det M_{\mathbf{f}}$ . The affine open subsets  $\overline{U}_{\mathbf{f}}$  cover  $\mathbf{H}_{\mathcal{A}}^{[n]}$ .

Recall that for every  $f(x_1, \dots, x_m) \in \mathcal{A}$  the element  $\mathbf{e} \bullet f(A_1, \dots, A_m)Y \in M_0$  is  $\mathrm{GL}_n$ -invariant and hence descends to a section of  $\mathcal{M}$ . Moreover  $\mathcal{M}$  is generated by sections of this form. Let us define a trivialization

$$\mathcal{M}|_{\overline{U}_{\mathbf{f}}} \xrightarrow{\sim} \mathcal{O}_{\overline{U}_{\mathbf{f}}}^{\oplus n}, \quad \mathbf{e} \bullet f(A_1, \dots, A_m)Y \mapsto M_{\mathbf{f}}^{-1} f(A_1, \dots, A_m)Y$$

where on RHS we have the product of the matrix  $M_{\mathbf{f}}^{-1}$  and the column vector  $f(A_1, \dots, A_m)Y$ . Note that this product consists of  $\mathrm{GL}_n$ -invariant functions and hence is indeed an element of  $\mathcal{O}_{\overline{U}_{\mathbf{f}}}^{\oplus n}$ . Therefore the cohomology class  $\Delta$  is represented by the 1-cocycle

$$\{M_{\mathbf{f}'}^{-1} \cdot M_{\mathbf{f}} : \overline{U}_{\mathbf{f}} \cap \overline{U}_{\mathbf{f}'} \rightarrow \mathrm{GL}_n\}$$



Consider the line bundle  $\mathcal{L} = \bigwedge^n \mathcal{M}$ . It defines the cohomology class  $\delta \in H^1(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathbb{G}_m)$  which is the image of  $\Delta$  under the map  $H^1(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathrm{GL}_n) \rightarrow H^1(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathbb{G}_m)$  induced by the homomorphism of  $k$ -group schemes  $\det : \mathrm{GL}_n \rightarrow \mathbb{G}_m$ . The class  $\delta$  is described by the cocycle

$$\left\{ \frac{D_{\mathbf{f}}}{D_{\mathbf{f}'}} : \overline{U}_{\mathbf{f}} \cap \overline{U}_{\mathbf{f}'} \rightarrow \mathbb{G}_m \right\}$$

For each  $p \geq 0$  define the  $k$ -module

$$T_p = \{h \in \Gamma(\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}, \mathcal{O}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}}) \mid \forall g \in \mathrm{GL}_n, g(h) = (\det g)^p h\}$$

In particular  $T_0 = \Gamma(\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}, \mathcal{O}_{\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}})^{\mathrm{GL}_n} = \Gamma(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{O}_{\mathbf{H}_{\mathcal{A}}^{[n]}})$ . Also notice that for each  $\mathbf{f} \in \mathcal{A}$  the function  $D_{\mathbf{f}} \in T_1$ . Let  $T = \bigoplus_{p \geq 0} T_p$  be the corresponding graded  $k$ -algebra.

**Lemma 3.1.** *There is a natural isomorphism of graded  $k$ -algebras*

$$T_{\bullet} \simeq \bigoplus_{p \geq 0} H^0(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{L}^{\otimes p}).$$

*In particular the functions  $D_{\mathbf{f}}$  correspond to sections of  $\mathcal{L}$ .*

*Proof.* Let  $h \in T_p$ . For each  $\mathbf{f}$  the function  $h/D_{\mathbf{f}}^p$  on  $U_{\mathbf{f}}$  is  $\mathrm{GL}_n$ -invariant, so it descends to a function on  $\overline{U}_{\mathbf{f}}$ . Clearly the collection  $\{h/D_{\mathbf{f}}^p \in \mathcal{O}(\overline{U}_{\mathbf{f}})\}$  defines a global section of  $\mathcal{L}^{\otimes p}$  and this induces an injective homomorphism of graded algebras

$$\theta : T_{\bullet} \rightarrow \bigoplus_{p \geq 0} H^0(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{L}^{\otimes p}).$$

Let us show the surjectivity of  $\theta$ . Let  $\gamma \in \Gamma(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{L}^{\otimes p})$ . Then  $\gamma$  corresponds to a collection of sections  $\{\gamma_{\mathbf{f}} \in \mathcal{O}(\overline{U}_{\mathbf{f}})\}$  such that  $\gamma_{\mathbf{f}'} = \frac{D_{\mathbf{f}}^p}{D_{\mathbf{f}'}^p} \gamma_{\mathbf{f}}$  on  $\overline{U}_{\mathbf{f}} \cap \overline{U}_{\mathbf{f}'}$ . This gives us a global function  $h$  on  $\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]}$  such that  $h|_{U_{\mathbf{f}}} = \pi^* \gamma_{\mathbf{f}} D_{\mathbf{f}}^p$ . Clearly  $h \in T_p$  and  $\theta(h) = \gamma$ .  $\square$

The line bundle has a description in terms of the universal bundle  $\mathcal{M}$  on  $\mathbf{H}_{\mathcal{A}}^{[n]}$ .

**Lemma 3.2.** *Let  $\mathcal{B}$  be another associative  $k$ -algebra and  $\mathcal{A} \rightarrow \mathcal{B}$  be a surjection. Denote by  $j : \mathbf{H}_{\mathcal{B}}^{[n]} \hookrightarrow \mathbf{H}_{\mathcal{A}}^{[n]}$  be the induces closed embedding. Let  $\mathcal{M}^{\mathcal{A}}$  and  $\mathcal{M}^{\mathcal{B}}$  (resp.  $\mathcal{L}^{\mathcal{A}}$  and  $\mathcal{L}^{\mathcal{B}}$ ) be the corresponding universal bundles (resp. line bundles) on  $\mathbf{H}_{\mathcal{A}}^{[n]}$  and  $\mathbf{H}_{\mathcal{B}}^{[n]}$  respectively. Then  $j^* M_u^{\mathcal{A}} = M_u^{\mathcal{B}}$  and  $j^* \mathcal{L}^{\mathcal{A}} = \mathcal{L}^{\mathcal{B}}$ .*

*Proof.* This follows from the commutativity of the natural diagram

$$\begin{array}{ccc} \tilde{\mathbf{H}}_{\mathcal{B}}^{[n]} & \hookrightarrow & \tilde{\mathbf{H}}_{\mathcal{A}}^{[n]} \\ \downarrow & & \downarrow \\ \mathbf{H}_{\mathcal{B}}^{[n]} & \xrightarrow{j} & \mathbf{H}_{\mathcal{A}}^{[n]} \end{array}$$

and the definition of the universal bundle  $\mathcal{M}$ . □

**Theorem 3.3.** *Assume that the ground ring  $k$  is a UFD and a finitely generated algebra over a field, and let  $F_m = k\langle x_1, \dots, x_m \rangle$  be the free associative algebra. Fix  $n \geq 1$  and consider the graded  $k$ -algebra  $S_{\bullet} = S_{\bullet}^{F_m} = \bigoplus_{p \geq 0} H^0(\mathbf{H}_{F_m}^{[n]}, \mathcal{L}^{\otimes p})$ . Then*

(i) *The graded  $k$ -algebra  $S_{\bullet}$  is finitely generated. In particular the  $k$ -algebra  $\mathcal{O}(\mathbf{H}_{F_m}^{[n]}) = S_0$  is finitely generated.*

(ii) *The line bundle  $\mathcal{L}$  on  $\mathbf{H}_{F_m}^{[n]}$  is ample, i.e. for some  $n_0 > 0$  the sections  $H^0(\mathbf{H}_{F_m}^{[n]}, \mathcal{L}^{\otimes n_0})$  define a closed embedding*

$$\mathbf{H}_{F_m}^{[n]} \hookrightarrow \mathbf{P}_{\mathcal{O}(\mathbf{H}_{F_m}^{[n]})}^N$$

for some  $N \geq 0$ .

*Proof.* (i) We will denote  $\mathbf{H}_{F_m}^{[n]}$  simply by  $\mathbf{H}^{[n]}$ , etc. In case  $n = 1$ ,  $\mathbf{H}^{[1]} = \text{Spec}k[t^1, \dots, t^m]$  and the line bundle  $\mathcal{L}$  is trivial (Example 2.9), hence the theorem holds with  $N = 0$ . So we may assume that  $n \geq 2$ .

Also let us first consider the case  $m = 1$ , so that  $F_1 = k[x]$  - the commutative polynomial ring over  $k$ . Choose  $f_1 = x, f_2 = x^2, \dots, f_{n-1} = x^{n-1} \in F_1$  and consider the corresponding affine open subset  $U_{x \dots x^{n-1}} \subset W \times V$ .

**Lemma 3.4.** *We have the equality  $U_{x \dots x^{n-1}} = \tilde{\mathbf{H}}^{[n]}$ .*

*Proof.* Indeed, by definition  $U_{x \dots x^{n-1}} \subset \tilde{\mathbf{H}}^{[n]}$  and both are open subschemes in  $W \times V$ . Hence it suffices to show that they have the same points. This is a consequence of the following simple observation: let  $\mathbf{F}$  be a field,  $M$  - an  $n$ -dimensional  $\mathbf{F}$ -vector space,  $m \in M$  and  $B \in \text{End}_{\mathbf{F}}(M)$ . We consider  $M$  as a  $\mathbf{F}[B]$ -module in the obvious way. Then  $\mathbf{F}[B]m = M$  if and only if the  $\mathbf{F}$ -span of  $m, Bm, \dots, B^{n-1}m$  is equal to  $M$ . □

It follows that for  $m = 1$ , we have  $\tilde{\mathbf{H}}^{[n]} = \text{Gl}_n \times \mathbf{H}^{[n]}$  and  $\mathbf{H}^{[n]}$  is an affine scheme. Also the line bundle  $\mathcal{L}$  on  $\mathbf{H}^{[n]}$  is trivial, hence the theorem holds with  $N = 0$ . So we may assume that  $m \geq 2$ .

**Lemma 3.5.** *If  $n, m \geq 2$  the restriction map  $\Gamma(W \times V, \mathcal{O}_{W \times V}) \rightarrow \Gamma(\tilde{\mathbf{H}}^{[n]}, \mathcal{O}_{\tilde{\mathbf{H}}^{[n]}})$  is an isomorphism.*

*Proof.* The ring  $\Gamma(W \times V, \mathcal{O}_{W \times V}) = k[t_{ij}^s, y_l]$  is a UFD. Recall that the scheme  $\tilde{\mathbf{H}}^{[n]}$  is the union of affine open subsets  $U_{f_1 \dots f_{n-1}}$  defined by inverting the polynomials  $D_{f_1 \dots f_{n-1}}$ . We can find two sets of elements  $\{f_1, \dots, f_{n-1}\}, \{f'_1, \dots, f'_{n-1}\} \subset F_m$  such that the corresponding polynomials  $D_{\mathbf{f}}, D_{\mathbf{f}'}$  are not equal to zero and are not proportional. It follows that any element in the field of fractions of  $\Gamma(W \times V, \mathcal{O}_{W \times V})$  which is a regular function on  $U$  and  $U'$  must be a polynomial. This proves the lemma.  $\square$

Now we can prove that the algebra  $S_\bullet$  is finitely generated. By Lemma 3.1 there is an isomorphism of graded  $k$ -algebras  $S_\bullet \simeq T_\bullet$ , where

$$T_p = \{h \in \Gamma(\tilde{\mathbf{H}}^{[n]}, \mathcal{O}_{\tilde{\mathbf{H}}^{[n]}}) \mid \forall g \in \mathrm{GL}_n, g(h) = (\det g)^p h\}$$

By Lemma 3.5

$$T_p = \{h \in k[t_{ij}^s, y_l] \mid \forall g \in \mathrm{GL}_n, g(h) = (\det g)^p h\}$$

Let  $\mathrm{GL}_n$  act on  $\mathrm{Speck}[t]$  by the formula  $g(t) = \det(g)^{-1}t$ . Consider the affine space  $W \times V \times \mathrm{Speck}[t]$  with the diagonal  $\mathrm{GL}_n$ -action. Let  $B$  denote the (polynomial) algebra of global sections on  $W \times V \times \mathrm{Speck}[t]$ . Since the group scheme  $\mathrm{GL}_n$  is reductive and  $k$  is of finite type over a field we know by [Se], Thm.2(i) that the  $k$ -algebra of invariants  $B^{\mathrm{GL}_n}$  is finitely generated. Note that the algebra  $B^{\mathrm{GL}_n}$  is graded

$$B^{\mathrm{GL}_n} = \bigoplus_{p \geq 0} T_p \cdot t^p.$$

Hence the graded algebra  $B^{\mathrm{GL}_n}$  is isomorphic to  $T_\bullet \simeq S_\bullet$ . In particular  $S_\bullet$  is also finitely generated.

**Lemma 3.6.** *Let  $k$  be a commutative ring and  $A_\bullet = \bigoplus_{p \geq 0} A_p$  be a finitely generated graded  $k$ -algebra. Then for some  $m > 0$  its graded  $A_0$ -subalgebra  $\bigoplus_{p \geq 0} A_{pm}$  is generated by  $A_m$ .*

*Proof.* Let  $x_1, \dots, x_n$  be generators of the  $A_0$ -algebra  $A_\bullet$  of degrees  $d_1, \dots, d_n$ , and let  $m = 2nD$ , where  $D = d_1 \cdots d_n$ . We claim that  $A_m$  generates the  $A_0$ -algebra  $\sum_p A_{pm}$ . Indeed, let  $X = x_1^{a_1} \cdots x_n^{a_n}$  denote an element of  $A_{pm}$ . Thus,  $pm = \sum a_i d_i$ . We may assume  $p \geq 2$ , since otherwise there is nothing to show.

Let  $y_j = x_j^{D/d_j}$ , so all  $y_j$  have degree  $D$ . We claim that there exists an  $n$ -tuple of non-negative integers  $b_j$  such that  $b_j D/d_j \leq a_j$  for all  $j$  and  $b_1 + \dots + b_n = 2n$ . If true, then  $X$  can be written as a product of  $y_1^{b_1} \dots y_n^{b_n} \in A_m$  and a monomial in the  $x_i$ 's, and we win by induction on  $p$ . To prove that the  $b_i$  exist, it suffices to show that  $\sum \lfloor a_j d_j / D \rfloor \geq 2n$ . However,

$$\sum_j \lfloor a_j d_j / D \rfloor > \sum_j (a_j d_j / D - 1) = pm/D - n \geq 2m/D - n = 3n > 2n.$$

□

(ii) We now prove that the line bundle  $\mathcal{L}$  is ample.

Recall that for each  $\mathbf{f} = \{f_1, \dots, f_{n-1}\} \in \mathcal{A}$  the function  $D_{\mathbf{f}}$  on  $\tilde{\mathbf{H}}^{[n]}$  belongs to  $T_1 \simeq S_1$ , hence defines a global section of  $\mathcal{L}$ . We can choose a finite set  $\{\mathbf{f}_i\}$ , say such that the affine open subsets  $\overline{U}_{\mathbf{f}_i} = \{D_{\mathbf{f}_i} \neq 0\}$  cover  $\mathbf{H}^{[n]}$ . Choose  $n_0 > 0$  as in the above Lemma 3.6. Suppose that the  $S_0$ -module  $H^0(\mathbf{H}^{[n]}, \mathcal{L}^{\otimes n_0})$  is generated elements  $\{s_0, \dots, s_N\}$  and consider the corresponding morphism

$$\phi : \mathbf{H}^{[n]} \rightarrow \mathbf{P}_{S_0}^N$$

defined by  $H^0(\mathbf{H}^{[n]}, \mathcal{L}^{\otimes n_0})$ . Notice that for each  $i$ ,  $D_{\mathbf{f}_i}^{n_0} \in H^0(\mathbf{H}^{[n]}, \mathcal{L}^{\otimes n_0})$ , hence the morphism  $\phi$  is well defined. We claim that  $\phi$  is a closed embedding.

**Lemma 3.7.** *Let  $R$  be a commutative ring and  $\phi : X \rightarrow \mathbf{P}_R^N$  be a morphism of  $R$ -schemes corresponding to an invertible sheaf  $L$  on  $X$  and sections  $s_0, \dots, s_N \in \Gamma(X, L)$ . Suppose that there exists a subset  $\{t_i\} \subset \{s_j\}$  such that*

(i) *each open subset  $X_{t_i} = \{t_i \neq 0\} \subset X$  is affine;*

(ii) *for each  $i$ , the map  $R[y_0, \dots, y_N] \rightarrow \Gamma(X_{t_i}, \mathcal{O}_{X_{t_i}})$  defined by  $y_j \mapsto s_j/t_i$  is surjective.*

*Then  $\phi$  is a closed embedding.*

*Proof.* This lemma is just a slight variation of [Ha], II, Proposition 7.2, and the proof is the same. □

To apply this lemma we may assume that the sections  $D_{\mathbf{f}_i}^{n_0}$ 's are among the  $s_j$ 's. Pick one such section  $D_{\mathbf{f}}^{n_0}$ . As explained above the corresponding open subset  $\overline{U}_{\mathbf{f}} \subset \mathbf{H}^{[n]}$  is affine. Choose  $\rho \in \mathcal{O}(\overline{U}_{\mathbf{f}})$ . Then  $\pi^{-1}(\rho) \in \mathcal{O}(U_{\mathbf{f}})$ , and hence  $\pi^{-1}(\rho) = h/D_{\mathbf{f}}^p$  for some  $h \in \Gamma(\tilde{\mathbf{H}}^{[n]}, \mathcal{O}_{\tilde{\mathbf{H}}^{[n]}})$  and  $p \geq 0$ . We may assume that  $p = n_0 p'$  and so

$h \in H^0(\mathbf{H}^{[n]}, \mathcal{L}^{\otimes n_0 p'})$  (since  $\pi^{-1}(\rho)$  is  $\mathrm{GL}_n$ -invariant). But then  $h$  is a polynomial in  $s_j$ 's with coefficients in  $S_0$ . This shows that the map  $S_0[s_0, \dots, s_N] \rightarrow \mathcal{O}(\overline{U}_{\mathbf{f}})$  defined by  $s_j \mapsto s_j/D_{\mathbf{f}}^{n_0}$  is surjective. Thus  $\phi$  is a closed embedding. This proves (ii) and the theorem.  $\square$

**Corollary 3.8.** *Let  $k$  be as in Theorem 3.3 and let  $\mathcal{A}$  be a finitely generated associative  $k$ -algebra, say  $\mathcal{A}$  is a quotient of  $F_m$ . Fix  $n \geq 1$ . Then for some  $n_0 > 0$  the sections  $H^0(\mathbf{H}_{\mathcal{A}}^{[n]}, \mathcal{L}^{\otimes n_0})$  define a closed embedding*

$$\mathbf{H}_{\mathcal{A}}^{[n]} \hookrightarrow \mathbf{P}_{\mathcal{O}(\mathbf{H}_{F_m}^{[n]})}^N$$

for some  $N \geq 0$ .

*Proof.* Indeed, by Remark 2.11 the scheme  $\mathbf{H}_{\mathcal{A}}^{[n]}$  is a closed subscheme in  $\mathbf{H}_{F_m}^{[n]}$  and the line bundle  $\mathcal{L}^{\mathcal{A}}$  is the restriction of  $\mathcal{L}^{F_m}$  (Lemma 3.2). Hence it remains to apply part (ii) of Theorem 3.3.  $\square$

**Remark 3.9.** *For a general  $\mathcal{A}$  as in Corollary 3.8 we cannot prove that the  $k$ -algebra  $\mathcal{O}(\mathbf{H}_{\mathcal{A}}^{[n]})$  (or the  $k$ -algebra  $\mathcal{O}(\tilde{\mathbf{H}}_{\mathcal{A}}^{[n]})$ ) is finitely generated. This is the reason for the appearance of the projective space  $\mathbf{P}_{\mathcal{O}(\mathbf{H}_{F_m}^{[n]})}^N$  in that corollary.*

**3.1. The tangent sheaf of the Hilbert scheme.** Assume that the base ring  $k$  is an algebraically closed field and let  $\mathcal{A}$  be a finitely generated  $k$ -algebra. All schemes and morphism of schemes are assumed to be defined over  $k$ . Fix  $n \geq 1$ . Put  $\mathbf{H} = \mathbf{H}_{\mathcal{A}}^{[n]}$  and consider the canonical short exact sequence of  $\mathcal{A}_{\mathbf{H}}$ -modules

$$(3.1) \quad 0 \rightarrow \mathcal{I} \rightarrow \mathcal{A}_{\mathbf{H}} \rightarrow \mathcal{M} \rightarrow 0$$

where  $\mathcal{M}$  is the universal bundle and  $\mathcal{I}$  is the "universal left ideal of codimension  $n$  in  $\mathcal{A}$ ". In particular this is a short exact sequence of quasi-coherent sheaves on  $\mathbf{H}$ . Since the  $\mathcal{O}_{\mathbf{H}}$ -module  $\mathcal{M}$  is locally free this a locally split sequence of quasi-coherent sheaves.

**Proposition 3.10.** *Let  $Z$  be a scheme and  $j : Z \rightarrow \mathbf{H}$  be a morphism of schemes. Then there is a natural isomorphism of two  $\mathcal{O}(Z)$ -modules:*

- 1) *The set  $T(Z)$  of morphisms  $\nu : Z \times \mathrm{Spec}k[\epsilon]/(\epsilon^2) \rightarrow \mathbf{H}$  such that the composition of the closed embedding  $Z \rightarrow Z \times \mathrm{Spec}k[\epsilon]/(\epsilon^2)$  with  $\nu$  coincides with  $j$ .*
- 2) *The set of morphisms  $\mathrm{Hom}_{\mathcal{A}_Z}(j^*\mathcal{I}, j^*\mathcal{M})$ .*

In particular, for a closed point  $x \in \mathbf{H}$  let  $\mathcal{M}_x$  be the corresponding quotient of  $\mathcal{A}$  by the left ideal  $\mathcal{I}_x \subset \mathcal{A}$ . Then the  $k$ -vector space  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{I}_x, \mathcal{M}_x)$  is naturally isomorphic to the Zariski tangent space of  $\mathbf{H}$  at  $x$ .

*Proof.* Put  $\Lambda := \mathrm{Spec}k[\epsilon]/(\epsilon^2)$  and let  $i : Z \rightarrow Z \times \Lambda$  and  $p : Z \times \Lambda \rightarrow Z$  be the closed embedding and the projection. Choose  $\phi \in T(Z)$ . We obtain two morphisms

$$jp, \phi : Z \times \Lambda \rightarrow \mathbf{H}$$

which induce the following diagram of  $\mathcal{A}_{Z \times \Lambda}$ -modules

$$\begin{array}{ccccc} & & \phi^* \mathcal{I} & & \\ & & \downarrow & & \\ (jp)^* \mathcal{I} & \xrightarrow{\alpha} & \mathcal{A}_{U \times \Lambda} & \rightarrow & (jp)^* \mathcal{M} \\ & & \downarrow \beta & & \\ & & \phi^* \mathcal{M} & & \end{array}$$

in which the horizontal (resp. the vertical) part is the pullback by  $jp$  (resp.  $\phi$ ) of the universal sequence 3.1. Since  $\phi i = j$  the image of the composition  $\beta \alpha$  is contained in  $\epsilon \phi^* \mathcal{M} = j^* \mathcal{M}$ . Hence this morphism factors through the quotient  $(jp)^* \mathcal{I} / \epsilon(jp)^* \mathcal{I} = j^* \mathcal{I}$ . So we obtain an element  $\beta \alpha \in \mathrm{Hom}_{\mathcal{A}_Z}(j^* \mathcal{I}, j^* \mathcal{M})$ .

Vice versa, let  $\gamma \in \mathrm{Hom}_{\mathcal{A}_Z}(j^* \mathcal{I}, j^* \mathcal{M})$ . Let  $U \subset Z$  be an affine open subset. We can choose a lift  $\tilde{\gamma} : (jp)^* \mathcal{I}|_{U \times \Lambda} \rightarrow \mathcal{A}_{U \times \Lambda}$  of  $p^* \gamma$  as a map of  $\mathcal{O}_{U \times \Lambda}$ -modules. Consider the ideal  $I_U \subset \mathcal{A}_{U \times \Lambda}$  generated by elements  $a - \epsilon \tilde{\gamma}(a)$  for  $a \in (jp)^* \mathcal{I}|_{U \times \Lambda}$ . This ideal  $I_U$  depends only on  $\gamma$  and not on a choice of  $\tilde{\gamma}$ . Hence we obtain a global ideal  $I \subset \mathcal{A}_{Z \times \Lambda}$ . Put  $M = \mathcal{A}_{Z \times \Lambda} / I$ . We have by construction  $i^* I = j^* \mathcal{I}$  as ideals in  $\mathcal{A}_Z$ . Hence also  $i^* M = j^* \mathcal{M}$  as  $\mathcal{A}_Z$ -modules. Notice also that  $I \cap \epsilon \mathcal{A}_{Z \times \Lambda} = \epsilon j^* \mathcal{I}$ , so that the  $\mathcal{O}_{Z \times \Lambda}$ -module  $M$  is locally free. In particular  $M \in M_{\mathcal{A}}^{[n]}(Z \times \Lambda)$ , so that  $M = \phi^* \mathcal{M}$  for a unique map  $\phi : Z \times \Lambda \rightarrow \mathbf{H}$ . This map  $\phi$  is in  $T(Z)$ , which defines the inverse map  $\mathrm{Hom}_{\mathcal{A}_Z}(j^* \mathcal{I}, j^* \mathcal{M}) \rightarrow T(Z)$ . So we have established a natural bijection of sets

$$T(Z) = \mathrm{Hom}_{\mathcal{A}_Z}(j^* \mathcal{I}, j^* \mathcal{M})$$

One can check that this is a bijection of  $\mathcal{O}(Z)$ -modules. This proves the proposition.  $\square$

Let

$$T_{\mathbf{H}} := \mathrm{Hom}_{\mathcal{O}_{\mathbf{H}}}(\Omega_{\mathbf{H}}^1, \mathcal{O}_{\mathbf{H}})$$

be the tangent sheaf of  $\mathbf{H}$ . For any open subset  $U \subset \mathbf{H}$  the sections  $T_{\mathbf{H}}(U)$  are the  $k$ -linear derivations of the sheaf  $\mathcal{O}_U$ . Since  $\mathbf{H}$  is of finite type over  $k$  it follows that  $T_{\mathbf{H}}$  is a coherent sheaf [EGA], Corollaire 16.5.6. The following is a variation on the Proposition 3.10.

**Proposition 3.11.** *There is a natural isomorphism of  $\mathcal{O}_{\mathbf{H}}$ -modules  $T_{\mathbf{H}} = \mathcal{H}om_{\mathcal{A}_{\mathbf{H}}}(\mathcal{I}, \mathcal{M})$ .*

*Proof.* Choose an affine open subset  $U \subset \mathbf{H}$ . It suffices to construct a natural isomorphism of  $H^0(U, \mathcal{O}_U)$ -modules

$$H^0(U, T_U) = \text{Hom}_{\mathcal{A}_U}(\mathcal{I}_U, \mathcal{M}_U)$$

**Lemma 3.12.** *Let  $Y$  be a  $k$ -scheme. Put  $\Lambda = \text{Spec}k[\epsilon]/(\epsilon^2)$  and consider the closed embedding  $i : Y \hookrightarrow Y \times \Lambda$  induces by the inclusion of  $\text{Spec}k$  in  $\Lambda$ . Then there is a natural bijection between the set of  $k$ -linear derivations  $\delta : \mathcal{O}_Y \rightarrow \mathcal{O}_Y$  and the set  $S(Y)$  of morphisms of  $k$ -schemes  $\mu : Y \times \Lambda \rightarrow Y$ , such that  $\mu i = \text{id}_Y$ . This bijection respects the  $\mathcal{O}(Y)$ -module structure.*

*Proof.* Exercise. □

We apply the lemma to the case  $Y = U$ . Since  $U$  is open in  $\mathbf{H}$  the set  $S(U) = H^0(U, T_U)$  as in the lemma coincides with the set  $T(U)$  of morphisms  $\nu : U \times \Lambda \rightarrow \mathbf{H}$  such that the composition  $\nu i : U \rightarrow \mathbf{H}$  coincides with the embedding  $U \hookrightarrow \mathbf{H}$ . But by Proposition 3.10 the  $\mathcal{O}(U)$ -module  $T(U)$  is isomorphic to the  $\mathcal{O}(U)$ -module  $\text{Hom}_{\mathcal{A}_U}(\mathcal{I}_U, \mathcal{M}_U)$ . This proves the proposition. □

#### 4. SOME LEMMAS ON FREE GROUP ACTIONS

In this subsection we prove some general simple statements about schemes with a free group action and about equivariant sheaves on them. Surely, this is "well known to experts" but we could not locate a reference.

Fix a scheme  $S$ . In this section all schemes are  $S$ -schemes. If a group scheme  $G$  acts on a scheme  $X$  we denote by  $m, p : G \times X \rightarrow X$  the action and the projection morphisms respectively.

**Definition 4.1.** *Let  $X$  be a scheme with an action of a group scheme  $G$ . A morphism  $\pi : X \rightarrow \overline{X}$  is called a Zariski locally trivial (resp. trivial) principal  $G$ -bundle if*

$$1) \quad \pi \cdot m = \pi \cdot p,$$

2) there exists an open covering  $\{W_i\}$  of  $\overline{X}$  and for each  $i$  an isomorphism of  $G$ -schemes

$$G \times W_i \simeq \pi^{-1}(W_i),$$

where the  $G$ -action on  $G \times W_i$  is by left multiplication on the first factor (resp.  $G \times \overline{X} \simeq X$ ).

**Lemma 4.2.** *Assume that a group scheme  $G$  acts on a scheme  $X$ . Suppose that there exists a  $G$ -equivariant map  $\sigma : X \rightarrow G$  (where  $G$  acts on itself by left multiplication). Denote by  $Z = \sigma^{-1}(e) \subset X$  the closed subscheme, which is the preimage of the identity  $e \in G$ . Then there exists a  $G$ -equivariant isomorphism  $G \times Z \simeq X$  (so that the projection  $X \rightarrow Z$  is a trivial principal  $G$ -bundle). If in addition the scheme  $X$  is affine and  $G$  is flat and quasi-compact, then  $Z = \text{Spec}\Gamma(X, \mathcal{O}_X)^G$ .*

*Proof.* Let  $i : Z \rightarrow X$  be the inclusion. We claim that the  $G$ -equivariant map

$$\theta : G \times Z \xrightarrow{(\text{id}, i)} G \times X \xrightarrow{m} X$$

is an isomorphism.

Denote by  $\sigma^{-1} : X \rightarrow G$  the composition of  $\sigma$  with the inverse map  $(-)^{-1} : G \rightarrow G$ .

Consider the composition

$$h : X \xrightarrow{(\sigma^{-1}, \text{id})} G \times X \xrightarrow{m} X, \quad x \mapsto \sigma^{-1}(x)x.$$

Then  $\sigma \cdot h$  maps  $X$  to  $e \in G$ . Therefore  $h : X \rightarrow Z$ , hence we obtain the map  $X \xrightarrow{(\sigma, h)} G \times Z$ , which is the inverse of  $\theta$  :

$$\begin{aligned} x &\xrightarrow{(\sigma, h)} (\sigma(x), \sigma^{-1}(x)x) \xrightarrow{\theta} \sigma(x)\sigma^{-1}(x)x = x, \\ (g, z) &\xrightarrow{\theta} gz \xrightarrow{(\sigma, h)} (\sigma(gz), \sigma^{-1}(gz)gz) = (g\sigma(z), (g\sigma(z))^{-1}gz) = (g, z). \end{aligned}$$

So  $\theta$  is an isomorphism.

Let us prove the last assertion. Since the group scheme  $G$  is flat, it is automatically faithfully flat. Hence the projection morphism  $G \times Z \rightarrow Z$  is faithfully flat and quasi-compact. Thus it is a morphism of strict descent with respect to quasi-coherent sheaves. It is explained in the proof of Proposition 4.5 below that this implies the equality

$$Z = \text{Spec}\Gamma(X, \mathcal{O}_X)^G.$$

□



**Proposition 4.3.** *Let a flat affine group scheme  $G$  act on a separated scheme  $X$ . Assume that  $X$  has an affine open covering  $\{U_i\}$  with the following property:*

*For each  $i$ ,  $U_i$  is  $G$ -invariant and there exists a  $G$ -equivariant map  $\sigma_i : U_i \rightarrow G$ .*

*Then there exists a universal categorical quotient  $\pi : X \rightarrow Z$  which is a Zariski locally trivial principal  $G$ -bundle.*

*Proof.* Denote by  $A_i$  the ring such that  $U_i = \text{Spec} A_i$ . It follows from Lemma 4.2 that for each  $i$  there exists a  $G$ -equivariant isomorphism  $U_i \simeq G \times Z_i$ , where  $Z_i = \sigma_i^{-1}(e)$  is the (closed subscheme of  $U_i$ ) preimage of the identity  $e \in G$ . Moreover, there is a natural isomorphism  $Z_i = \text{Spec}(A_i^G)$ . For two indices  $i, j$  we can apply this lemma to the  $G$ -invariant affine open subscheme  $U_i \cap U_j$  and the restriction of each of the  $G$ -equivariant maps  $\sigma_i, \sigma_j$ . We conclude that the affine schemes  $Z_i \cap U_j$  and  $Z_j \cap U_i$  are canonically isomorphic to  $\text{Spec} \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})^G$ . Hence there is a canonical isomorphism  $\phi_{ij} : Z_i \cap U_j \xrightarrow{\sim} Z_j \cap U_i$ . Clearly, the isomorphisms  $\{\phi_{ij}\}$  satisfy the cocycle condition and hence there is a scheme  $Z$  which is obtained by gluing the affine schemes  $Z_i$  along the open subschemes  $Z_i \cap U_j$  using the isomorphisms  $\phi_{ij}$ .

We have the canonical morphism of schemes  $\pi : X \rightarrow Z$  which locally on each  $U_i$  is induced by the inclusion of rings  $A_i^G \hookrightarrow A_i$ . Clearly  $\pi \cdot m = \pi \cdot p$ . If we identify  $U_i = G \times Z_i$ , then the restriction of the map  $\pi$  to  $U_i$  is the projection on the second factor. Hence  $\pi$  is a Zariski locally trivial principal  $G$ -bundle.

Obviously the map  $\pi|_{U_i} : U_i \rightarrow Z_i$  is a universal categorical quotient. Hence the map  $\pi$  is also such. This proves the proposition.  $\square$

Next we consider equivariant sheaves on schemes with free group actions.

For a scheme  $Y$  we denote by  $\text{Mod}(Y)$  the category of  $\mathcal{O}_Y$ -modules and by  $\text{Qcoh}(Y) \subset \text{Mod}(Y)$  its full subcategory of quasi-coherent sheaves on  $Y$ .

Let  $G$  be a group scheme acting on a scheme  $X$ . Consider the diagram of schemes

$$(4.1) \quad G \times G \times X \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} G \times X \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xleftarrow{d_1} \end{array} X$$

where

$$\begin{aligned} d_0(g_1, \dots, g_n, x) &= (g_2, \dots, g_n, g_1^{-1}x), \\ d_i(g_1, \dots, g_n, x) &= (g_1, \dots, g_i g_{i+1}, \dots, g_n, x), \quad 1 \leq i \leq n-1, \\ d_n(g_1, \dots, g_n, x) &= (g_1, \dots, g_{n-1}, x), \end{aligned}$$

$$s_0(x) = (e, x).$$

(Here  $e : S \rightarrow G$  is the identity of  $G$ .)

Recall that a  $G$ -equivariant quasi-coherent sheaf on  $X$  is a pair  $(F, \theta)$ , where  $F \in \mathrm{Qcoh}(X)$  and  $\theta$  is an isomorphism

$$\theta : d_1^* F \xrightarrow{\sim} d_0^* F$$

satisfying the cocycle condition

$$d_0^* \theta \cdot d_2^* \theta = d_1^* \theta, \quad s_0^* \theta = \mathrm{id}_F.$$

A morphism of equivariant sheaves is a morphism of quasi-coherent sheaves  $F \rightarrow F'$  which commutes with  $\theta$ . Denote by

$\mathrm{Qcoh}_G(X)$  the category of  $G$ -equivariant quasi-coherent sheaves on  $X$ .

**Remark 4.4.** Let  $Y$  be another scheme and let  $\sigma : X \rightarrow Y$  be a morphism such that  $\sigma \cdot d_0 = \sigma \cdot d_1$  (equivalently  $\sigma \cdot m = \sigma \cdot p$ .) Then we obtain the functor  $\sigma^* : \mathrm{Qcoh}(Y) \rightarrow \mathrm{Qcoh}_G(X)$ ,  $A \mapsto (\sigma^* A, \mathrm{id})$ . There exists a natural functor  $\sigma_*^G : \mathrm{Qcoh}_G(X) \rightarrow \mathrm{Mod}(Y)$ , which is the right adjoint to  $\sigma^*$ . Namely for  $(F, \theta) \in \mathrm{Qcoh}_G(X)$  and an open subset  $U \subset Y$  put

$$\sigma_*^G(F, \theta)(U) = \Gamma(\sigma^{-1}(U), F)^G := \{s \in F(\sigma^{-1}(U)) \mid d_0^* s = \theta(d_1^* s)\}.$$

**Proposition 4.5.** Let  $G$  be a quasi-compact and flat group scheme acting on a scheme  $X$ . Let  $\pi : X \rightarrow \overline{X}$  be a Zariski locally trivial principal  $G$ -bundle (Definition 4.1). Then the functor  $\pi^* : \mathrm{Qcoh}(\overline{X}) \rightarrow \mathrm{Qcoh}_G(X)$  is an equivalence of categories. This equivalence preserves locally free sheaves.

*Proof.* Case 1. Assume that  $\pi$  is a trivial principal homogeneous  $G$ -bundle, i.e.  $X = G \times \overline{X}$ . In this case the proposition follows from the faithfully flat descent for quasi-coherent sheaves. Indeed, it is easy to see that in this case the diagram 4.1 above is isomorphic to one where all the arrows  $d_i$  become projections and then a  $G$ -equivariant sheaf  $(F, \theta)$  is the same as a descent data for  $F \in \mathrm{Qcoh}(X)$  relative to the faithfully flat and quasi-compact morphism  $\pi : X = G \times \overline{X} \rightarrow \overline{X}$ . So it remains to apply the corresponding theorem of Grothendieck [SGA], Expose VIII, Thm.1.1.

Case 2. This is the general case. We first show that the functor  $\sigma^* : \mathrm{Qcoh}(\overline{X}) \rightarrow \mathrm{Qcoh}_G(X)$  is full and faithful. Let  $A \in \mathrm{Qcoh}(\overline{X})$ . It suffices to show that the adjunction morphism  $\alpha : A \rightarrow \sigma_*^G \sigma^* A$  is an isomorphism. But this question is local

on  $\overline{X}$ , hence we may assume that  $\pi$  is a trivial principal  $G$ -bundle. Thus  $\alpha$  is an isomorphism by Case 1.

Now we can prove that  $\sigma^*$  is essentially surjective. Let  $(F, \theta) \in \text{Qcoh}_G(X)$  and choose an open covering  $\{W_i\}$  of  $\overline{X}$  such that

$$\pi|_{\pi^{-1}(W_i)} : \pi^{-1}(W_i) \rightarrow W_i$$

is a trivial principal  $G$ -bundle. Let  $(F_i, \theta_i) \in \text{Qcoh}_G(\pi^{-1}(W_i))$  be the restriction of  $(F, \theta)$  to  $\pi^{-1}(W_i)$ . By Case 1 there exists a object  $A_i \in \text{Qcoh}(W_i)$  and an isomorphism  $\beta_i : \pi^*(A_i) \xrightarrow{\sim} (F_i, \theta_i)$ . Also for each pair of indices  $(i, j)$  there exists a unique isomorphism

$$\phi_{ij} : A_i|_{W_i \cap W_j} \xrightarrow{\sim} A_j|_{W_i \cap W_j}$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi^*(A_i|_{W_i \cap W_j}) & \xrightarrow{\pi^* \phi_{ij}} & \pi^*(A_j|_{W_i \cap W_j}) \\ \beta_i|_{W_i \cap W_j} \downarrow & & \downarrow \beta_j|_{W_i \cap W_j} \\ (F_i, \theta_i)|_{W_i \cap W_j} & = & (F_j, \theta_j)|_{W_i \cap W_j} \end{array}$$

The uniqueness of  $\phi_{ij}$  means that the collection  $\{\phi_{ij}\}$  satisfies the cocycle condition, so that  $A_i^s$  come from a global  $A \in \text{Qcoh}(\overline{X})$ . This proves the proposition.  $\square$

## REFERENCES

- [Ar-Zh] M. Artin, J.J. Zhang, Algebras and Representation Theory 4: 305-394, 2001.
- [EGA] A. Grothendieck, EGA IV, Publ. Math. IHES, 32 (1967) 5-361.
- [En-Rei] J. Engel, M. Reineke, Smooth models of quiver moduli, Math. Z. 262 (2009), 4, 817-848, arXiv:0706.4306.
- [Ha] R. Hartshorne, Algebraic Geometry, Springer 2010.
- [LeB] L. Le Bruyn, Noncommutative compact manifolds constructed from quivers, AMA Algebra Montp. Announc. 1999, Paper 1, 5 pp.
- [LeB-Se] L. Le Bruyn, G. Seelinger, Fibers of generic Brauer-Severi schemes, J. Algebra 214 (1999), no. 1, 222-234.
- [Rei] M. Reineke, Cohomology of non-commutative Hilbert schemes, Algebr. Represent. Theory, 8 (2005), 541-561, math.AG/0306185.
- [Se] C. S. Seshadri, Geometric reductivity over arbitrary base, Advances in Math. 26, 225-274, (1977).
- [SGA] A. Grothendieck, SGA I, LNM 224, Springer (1971).
- [VdB] M. Van den Bergh, The Brauer-Severi scheme of the trace ring of generic matrices. Perspectives in ring theory (Antwerp, 1987), 333-338, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988.

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